Lagrange Instability of Geodesics in Curved Double Twisted Liquid Crystals

by

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Abstract

It is shown that curved and flat helical double twisted liquid crystal (DTLC) in blue phase, can be unstable (stable) depending of the sign, negative (positive) of sectional curvature, depending on the pitch of the helix of the nematic crystal. In both cases Cartan torsion is presented. It is also shown that the instability or stability depends on the value of the pitch of the helix in nematic crystals. Frank energy stability A similar result using the method of Frank energy stability in the twist of cholesteric liquid crystal was given by Kiselev and Sluckin [PRE 71(2005)], where the free twist number determines the equilibrium value of the cholesteric liquid crystals (CLC) pitch of the helix. As a final example we solve the geodesic equations in twisted nematics with variable pitch helix and non-constant torsion. This non-Riemannian geometrical approach, also seems to unify two recent analysis of cylindrical columns given by Santangelo et al [PRL 99,(2007)] and the curved crystal endowed with torsion, given by Vitelli et al [Proc Nat Acad Sci (2006)]. Stability of toroidal curved surfaces were also previously considered by Bowick et al [PRE 69,(2004)], as an example of curvature-induced defect. Investigation of the Lagrangean instability may be useful in the investigation of HIV viruses and proteins. **PACS numbers:**

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I Introduction

Following the steps of success of the applications of the differential geometry of curves and surfaces [1], through Riemannian geometry in Einstein theory of gravitation [2], as well as the success of the use of Cartan non-Riemannian geometry endowed with a torsion tensor, to explain alternatives theory of gravitation such as Einstein-Cartan gravity [3], many mathematical and crystal physicists, have applied these geometries to the explanation of disclination (Riemann curvature) or dislocation (non-Riemannian Cartan torsion), defects in crystals [4]. Even recently Riemannian geometry has been applied in the investigation of twisted magnetic flux tubes in plasmas helical flows MHD with and without vortices [5, 6, 7]. More recently these paths, two recent papers on the stability of geodesics in the curved and flat cylindrical liquid crystals in curved surface with defects [8] and columnar phases [9] have appeared in the literature. In the first paper Vitelli et al [8] have discussed the instability driven by a purely geometric potential in Gaussian curvature of the stress law covariant generalization. In the second Santangelo et al [9] have been considered the instability or stability of geodesics in the geometry of columnar phases of nematic liquid crystal, in the analogy with the focusing process of a lensing ray in optics or gravitational lensing. Recently we also use the sectional developed by Kambe [10] curvature method to investigate the Lagrangean instability of a Couette type flow, viscous and sheared. Actually Kambe [10] has previously investigate Couette planar flows stability, and showed that Riemann curvature tensor would exist in the case of existence of pressure, even a constant one. The stability of the Couette planar flow is also obtained by him, using the symmetry in the case of symmetric scalar stress potential and by making use of the technique of Riemann sectional curvature, where the negative sectional curvature indicates instability of the flow, in the Lagrangean sense. In this brief report this idea can be easily transported to investigate the instability of DTLC nematic blue phase, where as shown by Pansu and Dubois-Violette [11] the presence of torsion is taken from granted from the very beginning. As a last example we modify the constant torsion twisted nematic geometry with variable pitch and compute the geodesics for this CLC. Therefore we investigate the Lagrange instability in two simple examples, in analogy with the works of Vitelli et al and Santangelo et al, namely of a Riemann-flat torsioned DTLC , while in the second example

a curved crystal is considered. The usefulness of the investigation of the Riemannian and non-Riemannian geometries in physical allows one also building models for analog gravity as the case of non-Riemannian geometry of vortex acoustic flows [12]. Other curvature-induced defect in toroidal Riemann geometry were also investigated by Bowick et al [13]. Yet another example of the use of Gaussian curvature on continuum elastic model topological defect in Riemannian manifolds was investigated by Giomi and Bowick [14]. The paper is organized as follows: Section II presents a brief review of sectional Riemannian curvature method in the coordinate-free language which helps to unify previously examples of instability. Section III presents the examples of curved and flat DTLC with totally skew torsion Lagrange stability and the dependence of the instability on the value of the pitch of helix, a similar result obtained by Kiselev and Sluckin [15]. Section IV presents the computation of geodesics of twisted nematic geometry. In section V conclusions are presented.

II Sectional Riemannian curvature Instability

In this section, before we add we make a brief review of the differential geometry of surfaces in coordinate-free language. The Riemann curvature is defined by

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{II.1}$$

where $X \epsilon T\mathcal{M}$ is the vector representation which is defined on the tangent space $T\mathcal{M}$ to the manifold $\mathcal{M}$. Here $\nabla_X Y$ represents the covariant derivative given by

$$\nabla_X Y = (X.\nabla)Y \tag{II.2}$$

which for the physicists is intuitive, since we are saying that we are performing derivative along the X direction. The expression $[X,Y]$ represents the commutator, which on a vector basis frame $\vec{e}_l$ in this tangent sub-manifold defined by

$$X = X_k \vec{e}_k \tag{II.3}$$

or in the dual basis ∂_k

$$X = X^k \partial_k \tag{II.4}$$

can be expressed as

$$[X,Y]=(X,Y)^k\partial_k \tag{II.5}$$

In this same coordinate basis now we are able to write the curvature expression (II.1) as

$$R(X,Y)Z:=[R^l{}_{jkp}Z^jX^kY^p]\partial_l \tag{II.6}$$

where the Einstein summation convention of tensor calculus is used. The expression $R(X,Y)Y$ which we shall compute bellow is called Ricci curvature. The sectional curvature which is very useful in future computations is defined by

$$K^{Riem}(X,Y):=\frac{<R(X,Y)Y,X>}{S(X,Y)} \tag{II.7}$$

where $S(X,Y)$ is defined by

$$S(X,Y):=||X||^2||Y||^2-<X,Y>^2 \tag{II.8}$$

where the symbol $<,>$ implies internal product. In the non-Riemannian (NR) case, the torsion two-form $T(X,Y)$ is defined by

$$T(X,Y):=\frac{1}{2}[\bar{\nabla}_XY-\bar{\nabla}_YX-[X,Y]] \tag{II.9}$$

where $\bar{\nabla}$ is the non-Riemannian connection [7] endowed with torsion. As in EC theory [5] the geodesic equation does not depend on torsion; only Jacobi deviation equation depends on torsion which is enough for investigate the role of torsion on stability. Since the Jacobi equation is given by

$$\frac{d^2J}{ds^2}=[||\nabla_{\vec{t}}\vec{e}_J||^2-K^{NR}(t,\vec{e}_J)||J|| \tag{II.10}$$

where $||\vec{e}_J||=1$ and J is the Jacobi field, representing the separation between geodesics,while $\vec{t}$ is the geodesic tangent vector. Here $K^{NR}(X,Y)$ is given by

$$K^{NR}(X,Y)=K^{Riem}(X,Y)+2<T(X,Y),\bar{\nabla}_YX> \tag{II.11}$$

Here as we shall see bellow the geodesic equation is $\nabla_YY=0$ simplified this expression. Note from this expression that the instability, or separation of the geodesics in the flow

$$\frac{d^2J}{ds^2}\geq 0 \tag{II.12}$$

implies that $K^{Riem}<0$ which is the condition for Lagrange instabillity.

III Stability of Curved and flat Liquid Crystal with torsion

In this section let us address the examples of Riemann-flat TDLC and the curved LC. In the first case we consider the CLC, where twist of molecules exists in just one direction, which a ordered phase is given by the director field

$$n^x = 0 \tag{III.13}$$

$$n^y = cospx \tag{III.14}$$

$$n^z = sinpx \tag{III.15}$$

where $q := \frac{2\pi}{p}$ is the wave number and p is the pitch of the helix. In the blue phase local constraint of minimum energy is double twist , which physically means that twist occurs in more than one direction in space. In the Euclidean frame this equation is

$$\delta_k n^j + p\epsilon_{kip} n^i \delta^{jp} = 0 \tag{III.16}$$

In the Riemann-flat manifold, where the Riemann tensor $R(X,Y)Z$ vanishes, or in components

$$R^i{}_{jkl} := 0 \tag{III.17}$$

In the frame where the covariant derivative of the basis $(\vec{e}_i)$ vanishes or $\nabla_{\vec{e}_i}\vec{e}_j = 0$, where from expression (II.9) the Cartan torsion reads

$$T(\vec{e}_i, \vec{e}_j) = -[\vec{e}_i, \vec{e}_j] \tag{III.18}$$

The double twist Levi-Civita connection is

$$\nabla_k \vec{n}^j = \partial_k \vec{n}^j + \Gamma^j{}_{ik} \vec{n}^i = 0 \tag{III.19}$$

Now computing the general Riemann-Cartan curvature tensor

$$R^{RC}(\vec{e}_i, \vec{e}_j) = \nabla_{\vec{e}_i}\nabla_{\vec{e}_j} - \nabla_{\vec{e}_i}\nabla_{\vec{e}_i} - \nabla_{[\vec{e}_i,\vec{e}_j]} \tag{III.20}$$

Using the expression for the Cartan torsion 2-form $T(X,Y)$ in terms of the Lie bracket above, the RC curvature reduces to

$$R^{RC}(\vec{e}_i, \vec{e}_j)\vec{e}_j = -\nabla_{[\vec{e}_i,\vec{e}_j]}\vec{e}_j = ([\vec{e}_i, \vec{e}_j].\nabla)\vec{e}_j \tag{III.21}$$

therefore to compute this expression one computes the Lie bracket in terms of Cartan torsion as

$$T(\vec{e}_i, \vec{e}_j) = (\vec{e}_i.\nabla)\vec{e}_j - (\vec{e}_j.\nabla)\vec{e}_i = [\vec{e}_i, \vec{e}_j] = \partial_i \vec{e}_j - \partial_j \vec{e}_i = T^k{}_{ij}\vec{e}_k \tag{III.22}$$

substitution of these components of torsion into expression (III.21) yields

$$R^{RC}(\vec{e}_i, \vec{e}_j)\vec{e^j} = ([\vec{e}_i, \vec{e}_j].\nabla)\vec{e^j} = (T^k{}_{ij}\vec{e}_k.\nabla)\vec{e^j} = (T^k{}_{ij}\partial_k \vec{e^j}) = T^k{}_{ij}T^{lij}\vec{e}_l \tag{III.23}$$

Thus the cholesteric blue phase of liquid crystals possesses the sectional curvature

$$K(\vec{e}_i, \vec{e}_j) =< R(\vec{e}_i, \vec{e}_j)\vec{e}_j, \vec{e}_i >= T^{kij}T_{kij} = -p\epsilon_{ijk}\epsilon^{ijk} = -6p^2 < 0 \tag{III.24}$$

since this curvature is negative, the instability of the CLC is proved in this case. In a more general Riemannian manifold we shall prove now that depending on the value of the pitch the CLC can be stable in the Lagrangean sense. This can be done as follows. First we shall compute the Riemann tensor with the condition of geodesic $\nabla_Y Y = 0$ is given, which simplifies the curvature expression to

$$R(X,Y)Y = -\nabla_Y[\bar{\nabla}_X Y + pY \wedge X] + p(X \wedge Y.\nabla)Y \tag{III.25}$$

where now, we have used the covariant derivative

$$\nabla_X Y := \bar{\nabla}_X Y + pY \wedge X] \tag{III.26}$$

To simplifies computations we assume that the term $\bar{\nabla}_X Y$ vanishes which after some algebra leads to expression

$$R(X,Y)Y = [-p^2 + p]Y \wedge (X \wedge Y) \tag{III.27}$$

Finally from this expression we can write the expression for the sectional curvature as

$$K(X,Y) =< R(X,Y)Y, X >= [-p^2 + p] < Y \wedge (X \wedge Y), X >= [-p^2 + p]\alpha ||X||^2 \tag{III.28}$$

Thus this time the sectional derivative can be positive or the twisted nematic crystal can be geodesically stable, depending on the value of the helix pitch.

IV Geodesics of twisted nematic with variable pitch helix

Let us built the Riemann metric in analogy to Katanaev and Volovich [16] as the deformation of the crystals $e_{ij} = n_{(j,i)}$ where this deformation represents the perturbation on the metric g_{ij} given by

$$g_{ij} = \delta_{ij} + e_{ij} \tag{IV.29}$$

where δ_{ij} is the delta Kronecker symbol and the vector n_i where $(i = 1, 2, 3)$ is the director field of nematic liquid crystal. we consider here that Riemann curvature tensor vanishes as in Einstein theory of absolute parallelism, which implies that the anti-symmetric connection (torsion tensor) is given in terms of the metric by

$$T_{ijk} = g_{jk,i} - g_{ji,k} \tag{IV.30}$$

Here the comma denotes partial derivatives with respect to the lower index. Substitution of expression (IV.29) into (IV.30) yields the following torsion vector

$$T_k = [\nabla^2 n_k - \delta_k(divn)] \tag{IV.31}$$

therefore and the Weitzenböck condition for teleparallelism on the curvature Riemann tensor $R_{ijkl}(\Gamma) = 0$,where Γ is the Riemann-Cartan affine connection.Later on we shall make an application of this formula to a special case of nematic crystal.In the meantime let us compute the geodesics equations for the corresponding metric of the liquid crystal.The geodesics equations

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \tag{IV.32}$$

and

$$\ddot{x}^i + \delta^{ik} n_{k,lj}\dot{x}^l\dot{x}^j = 0 \tag{IV.33}$$

where $\Gamma^i_{jk} = \frac{1}{2}\delta^{il}[g_{lj,k} + g_{lk,j} - g_{jk,l}]$ and we use the Euclidean 3D metric δ_{ij} to raise and lower indices.Let us now apply these ideas to the pure twist geometry of the nematic liquid crystals,where the director field is now given by the components

$$n_z = cos\theta(y) \tag{IV.34}$$

and

$$n_x = sin\theta(y) \tag{IV.35}$$

Where θ is the twist angle and the planar crystal is orthogonal to the y-coordinate direction. Substituting expressions (IV.33) and (IV.34) into (IV.35) one obtains the following components of the torsion vector $T^i_{ki} = T_k$

$$T_y(\theta) = -\partial_y(divn) \tag{IV.36}$$

and

$$T_z(\theta) = -\nabla^2 n_z \tag{IV.37}$$

Since $divn = n_{x,x} + n_{y,y} + n_{z,z} = 0$, there is no torsion vector component along the orthogonal direction. Thus the only non-vanishing torsion component of torsion reads

$$T_z(\theta) = -[cos\theta(\frac{d\theta}{dy})^2 + sin\theta\frac{d^2\theta}{dy^2}] \tag{IV.38}$$

Since local equilibrium conditions on the Liquid crystals yields [11]

$$\frac{d\theta}{dy} = constant = K \tag{IV.39}$$

Thus equation (IV.38) reduces to

$$T_z(\theta) = -K^2 cos\theta \tag{IV.40}$$

This last expression tell us that the twisted geometry of the crystal leads to a 3D helical torsion.Geodesics of the twisted geometry of the liquid crystal leads to the results

$$\ddot{x} + n^x_{,yy}\dot{y}^2 = 0 \tag{IV.41}$$

and

$$\ddot{y} = 0 \tag{IV.42}$$

and finally

$$\ddot{z} + n^z_{,yy}\dot{y}^2 = 0 \tag{IV.43}$$

Here the dots means derivation with respect to time coordinate. A simple algebraic manipulation yields

$$\ddot{x} + {K_1}^2 sin\theta(y) = 0 \tag{IV.44}$$

and

$$\ddot{z} + {K_1}^2 cos\theta(y) = 0 \tag{IV.45}$$

To simplify matters let us solve these equations in the approximation of small twist angles $\theta <<< 0$ where above equations reduces to

$$\ddot{x} + {K_1}^2\theta = 0 \tag{IV.46}$$

and

$$\ddot{z} + {K_1}^2 = 0 \tag{IV.47}$$

Therefore from expression (IV.47) one obtains the following solution

$$z = -{K_1}^2 t + c \tag{IV.48}$$

where c is an integration constant.To solve the remaining equations we need an explicit form of the twist angle with respect to time ,this can be obtained from the integration of the expression $\frac{d\theta}{dy} = 0$ which yields

$$\theta = Ky + f \tag{IV.49}$$

where f is another integration constant.Substitution of $y = K_0 t + d$ into this equation yields

$$\theta = mt + g \tag{IV.50}$$

where f,m and g are new integration constants. Thus one obtains

$$\ddot{x} = -({K_1}^2 mt + K_1 g) \tag{IV.51}$$

Integration of this expression yields

$$x(t) = -\frac{1}{6}(\alpha t^3 + \beta t^2 + \gamma t + \delta) \tag{IV.52}$$

substitution of the expression $t = \frac{y}{K_0}$ into (IV.52) yields

$$x = -\frac{1}{6}(\alpha^{'} y^3 + \beta^{'} y^2 + \gamma^{'} y + \delta) \tag{IV.53}$$

where all the greek letters represent new integration constants.Therefore the trajectory of the test particles is represented by a third order polinomial curve.Trajectories of test particles in domain walls are in general parabolic curves.After we finish this letter we hear that Dubois-Violette and Pansu [11] have considered a similar application of teleparallelism to cholesteric Blue Phase of liquid crystals.Nevertheless in their paper Cartan torsion is constant

V Conclusions

One of the most physical question in liquid crystal physics is the investigation of the stability of flows in the Euclidean manifold $\mathcal{E}^3$ using the sectional curvature of geodesic deviation. In this paper we discuss and present the contribution of Cartan torsion tensor and its role in curved and Riemann flat liquid crystals in the blue phase. The Lagrange instabilities in several CLC cases are computed and it is shown that instability may depend upon the value of twisting number or the pitch of the helix of nematics. The models discussed here may also be useful in building the analog models of stability of geodesics in non-Riemannian theories of gravity.

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